Application of Optimal Homotopy Analysis Method for Solitary Wave Solutions of Kuramoto-Sivashinsky Equation

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In this paper, the optimal homotopy analysis method is applied to find the solitary wave solutions of the Kuramoto-Sivashinsky equation. With three auxiliary convergence-control parameters, whose possible optimal values can be obtained by minimizing the averaged residual error, the method used here provides us with a simple way to adjust and control the convergence region of the solution. Compared with the usual homotopy analysis method, the optimal method can be used to get much faster convergent series solutions.

Key words: Kuramoto-Sivashinsky Equation; Optimal Homotopy Analysis Method; Solitary Wave Solution.

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1. Introduction

In past decades, both mathematicians and physicists have devoted considerable effort to the study of explicit solutions of the partial differential equations. Many powerful methods have been presented, such as inverse scattering method [1], Bäcklund transformation [2], Darboux transformation [3], Lie symmetry method [4], Hirota method [5], etc. Along with the development of computer technology and symbolic-numerical computation software such as Matlab, Maple, Mathematica and so on, these methods exhibit powerful capabilities. Among them, the homotopy analysis method (HAM), which was firstly proposed by Liao [6], based on the idea of homotopy in topology, is a general analytic method for nonlinear problems. Unlike the traditional methods (for example, perturbation techniques and so on), the HAM contains auxiliary parameters which provide us with a simple way to adjust and control the convergence region and rate of convergence of the series solution and has been successfully employed to solve explicit analytic solutions for many types of nonlinear problems [7 – 15].

However, as illustrated in [15], the usual HAM has only one convergence-control parameter c_0 and it is a pity that curves for convergence-control parameter (i.e. c_0-curves) can not tell us which value of $c_0 \in \mathbb{R}$ gives the fastest convergent series. Recently, to overcome this shortcoming, Liao [15] proposed an optimal HAM with more than one convergence-control parameter. Liao also introduced the so called averaged residual error to get the possible optimal convergence-control parameters efficiently, which can give good approximations of the optimal convergence-control parameters of the exact residual error. In general, the optimal HAM can greatly modify the convergence of homotopy series solution.

The aim of this paper is to directly apply the optimal HAM to reconsider the solitary wave solutions of the Kuramoto-Sivashinsky equation. Three convergence-control parameters are used in the method to accelerate the convergence of homotopy series solution which can give much better approximations. The optimal convergence-control parameters have been determined by minimizing the averaged residual error. The results obtained here show that they convergence much faster than those given by the usual HAM.

2. Optimal HAM for the Kuramoto-Sivashinsky Equation

The Kuramoto-Sivashinsky equation

$$u_t + \alpha u u_x + \beta u_{2x} + k u_{4x} = 0, \quad (1)$$

where α, β, and k are arbitrary constants, usually describes the fluctuations of the position of a flame front, the motion of a fluid going down a vertical wall, or a spatially uniform oscillating chemical reaction in a homogeneous medium and has been the subject of extensive research work in recent publications [16–18]. For example, the solitary wave solutions of the Kuramoto-Sivashinsky equation has been found in [14] by HAM. In the following, we will apply the optimal HAM to the Kuramoto-Sivashinsky equation to reconsider the solitary wave solutions again.

According to [14], in order to find the solitary wave solutions of (1), it is convenient to introduce a new dependent variable $w(\xi)$ defined by

$$u(x,t) = aw(\xi), \tag{2}$$

where $\xi = x - ct$, a is the amplitude, and c is the wave speed. Substitution of u given by (2) into (1) gives

$$kw^{(4)} + \beta w'' + \alpha aww' - cw' = 0 \tag{3}$$

and integrating once gives

$$kw''' + \beta w' + \frac{\alpha}{2}aw^2 - cw = 0, \tag{4}$$

where the prime denotes the differentiation with respect to ξ. Write

$$w(\xi) \approx B\exp(-\mu\xi) \quad \text{as} \quad \xi \to \infty, \tag{5}$$

where $\mu > 0$ and B are constants. Substituting (5) into (4) and balancing the main term yields

$$k\mu^3 + \beta\mu = -c, \tag{6}$$

and we consider the smallest positive real value for μ. Writing $\eta = \mu\xi$, (4) becomes

$$k\mu^3 w''' + \beta\mu w' + \frac{\alpha}{2}aw^2 - cw = 0, \tag{7}$$

where the prime denotes the derivative with respect to η. Assume that the dimensionless wave solution $w(\eta)$ arrives its maximum at the origin. Obviously, $w(\eta)$ and its derivatives tend to zero when $\eta \to \infty$. Besides, due to the continuity, the first derivative of $w(\eta)$ at crest is zero. Thus, the boundary conditions of the solitary wave solutions are

$$w(0) = 1, \; w'(0) = 0, \; w(\infty) = 0. \tag{8}$$

According to (7) and the boundary conditions (8), the solitary wave solution can be expressed by

$$w(\eta) = \sum_{m=1}^{+\infty} d_m \mathrm{e}^{-m\eta}, \tag{9}$$

where d_m $(m = 1,2,\ldots)$ are coefficients to be determined. Moreover, according to the rule of solution expression denoted by (9) and the boundary conditions (8), it is natural to choose $w_0(\eta) = 2\mathrm{e}^{-\eta} - \mathrm{e}^{-2\eta}$ as the initial approximation of $w(\eta)$.

Let $p \in [0,1]$ denote the embedding parameter, $c_0 \neq 0$ an auxiliary parameter, called the convergence-control parameter, and $\phi(\eta;p)$ a kind of continuous mapping of $w(\eta)$, respectively, we can construct following generalized homotopy:

$$\begin{aligned}(1-C(p))\mathcal{L}[\phi(\eta;p) - w_0(\eta)] = \\ c_0 B(p)\mathcal{N}[\phi(\eta;p), A(p)],\end{aligned} \tag{10}$$

where

$$\mathcal{L}[\phi(\eta;p)] = \left(k\mu^3\frac{\partial^3}{\partial\eta^3} + \beta\mu\frac{\partial}{\partial\eta} - c\right)\phi(\eta;p) \tag{11}$$

is an auxiliary linear operator, with the property

$$\mathcal{L}\left[C_1\mathrm{e}^{-\eta} + C_2\mathrm{e}^{\kappa_1\eta} + C_3\mathrm{e}^{\kappa_2\eta}\right] = 0, \tag{12}$$

where C_1, C_2, and C_3 are constants and

$$\kappa_{1,2} = \frac{c + k\mu \pm \sqrt{-3c^2 - 2c\beta\mu + \beta^2\mu^2}}{2(c+k\mu)}, \tag{13}$$

which in most cases are not positive integers. From (7), we define the nonlinear operator

$$\begin{aligned}\mathcal{N}[\phi(\eta;p), A(p)] = k\mu^3\frac{\partial^3\phi}{\partial\eta^3} + \beta\mu\frac{\partial\phi}{\partial\eta} \\ + \frac{\alpha}{2}A(p)\phi^2 - c\phi.\end{aligned} \tag{14}$$

In (10), $B(p)$ and $C(p)$ are the so-called deformation functions satisfying

$$B(0) = C(0) = 0, \; B(1) = C(1) = 1, \tag{15}$$

whose Taylor series

$$B(p) = \sum_{m=1}^{+\infty} \nu_m p^m, \; C(p) = \sum_{m=1}^{+\infty} \sigma_m p^m \tag{16}$$

exist and are convergent for $|p| \leq 1$.

Then when $p = 0$, according to the definition of $\mathcal{L}$ and $w_0(\eta)$, it is obvious that $\phi(\eta;0) = w_0(\eta)$. When $p = 1$, according to the definition (14), (10) is equivalent to the original (7), provided $\phi(\eta;1) = w(\eta)$. Thus, as p increases from 0 to 1, the solution $\phi(\eta;p)$ varies (or deforms) continuously from the initial guess $w_0(\eta)$ to the solution $w(\eta)$ of (7).

According to [15], there are infinite numbers of deformation functions satisfying the properties (15) and (16). And in theory, the more the convergence-control parameters are used, the better approximation one should obtain by this generalized HAM. But for the sake of computation efficiency, we use here the following one-parameter deformation functions

$$B(c_1;p) = \sum_{m=1}^{+\infty} \nu_m(c_1)p^m, \; C(c_2;p) = \sum_{m=1}^{+\infty} \sigma_m(c_2)p^m, \tag{17}$$

where $|c_1| < 1$ and $|c_2| < 1$ are constants, which are convergence-control parameters too, and

$$\nu_1(c_1) = 1-c_1, \; \nu_m(c_1) = (1-c_1)c_1^{m-1}, \; m>1, \tag{18}$$

$$\sigma_1(c_2) = 1-c_2, \; \sigma_m(c_2) = (1-c_2)c_2^{m-1}, \; m>1. \tag{19}$$

The different values of c_1 give different paths of $B(c_1;p)$ as shown in Figure 1. Note that $B(c_1;p)$ and $C(c_2;p)$ contain the convergence-control parameters c_1 and c_2, respectively. So, we have at most three unknown convergence-control parameters c_0, c_1, and c_2, which can be used to ensure the convergence of solutions series, as shown later.

Then the so-called zeroth-order deformation equation becomes

$$\begin{aligned}(1-C(c_2;p))\mathcal{L}[\phi(\eta;p)-w_0(\eta)] = \\ c_0B(c_1;p)\mathcal{N}[\phi(\eta;p),A(p)],\end{aligned} \tag{20}$$

and according to (8), it should subject to following boundary conditions:

$$\begin{aligned}&\phi(0;p)=1, \quad \left.\frac{\partial\phi(\eta;p)}{\partial\eta}\right|_{\eta=0}=0,\\ &\phi(\infty;p)=0, \quad \left.\frac{\partial\phi(\eta;p)}{\partial\eta}\right|_{\eta=+\infty}=0.\end{aligned} \tag{21}$$

Obviously, $\phi(\eta;p)$ is determined by the auxiliary linear operator $\mathcal{L}$, the initial guess $w_0(\eta)$, and the convergence-control parameters c_0, c_1, and c_2. Note that we have great freedom to choose all of them. Assuming that all of them are so properly chosen that the

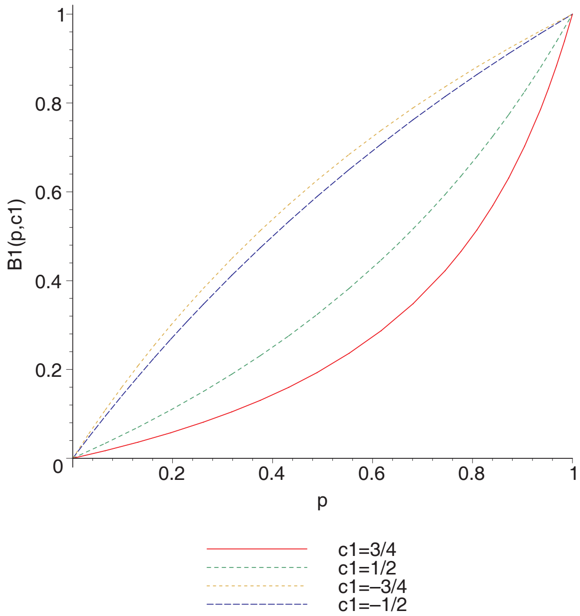

Fig. 1 (colour online). Deformation function $B_1(p;c_1)$ defined by (17) and (18). From top to bottom: yellow long-dashed line: $c_1=-3/4$; blue space-dash line: $c_1=-1/2$; green dotted line: $c_1=1/2$; red solid line: $c_1=3/4$.

Taylor series

$$\begin{aligned}\phi(\eta;p) &= w_0(\eta)+\sum_{m=1}^{+\infty} w_m(\eta)p^m,\\ A(p) &= a_0+\sum_{m=1}^{+\infty} a_m p^m\end{aligned} \tag{22}$$

exist and converge at $p=1$, we have the following homotopy series solution:

$$w(\eta) = w_0(\eta)+\sum_{m=1}^{+\infty} w_m(\eta), \quad a = a_0+\sum_{m=1}^{+\infty} a_m, \tag{23}$$

where

$$w_m(\eta) = \frac{1}{m!}\left.\frac{\partial^m\phi(\eta;p)}{\partial p^m}\right|_{p=0}, \quad a_m = \frac{1}{m!}\left.\frac{\partial^m A(p)}{\partial p^m}\right|_{p=0}. \tag{24}$$

Let G denote a function of $p\in[0,1]$ and define the so-called mth-order homotopy derivative [19]:

$$D_m[G] = \frac{1}{m!}\left.\frac{\partial^m G}{\partial p^m}\right|_{p=0}. \tag{25}$$

Taking above operator on both sides of the zeroth-order deformation equation (20) and the boundary conditions (21), we have the following mth-order deformation equation:

$$\mathcal{L}\left[w_m(\eta)-\chi_m\sum_{n=1}^{m-1}\sigma_{m-n}(c_2)w_n(\eta)\right]= c_0\sum_{n=0}^{m-1}\nu_{m-n}(c_1)R_n(\eta), \tag{26}$$

subject to the boundary conditions

$$w_m(0)=w_m'(0)=w_m(+\infty)=0, \tag{27}$$

where

$$R_n(\eta)=k\mu^3 w_n'''+\beta\mu w_n' +\frac{\alpha}{2}\sum_{i=0}^{n}\left(a_{n-i}\sum_{j=0}^{i}w_j w_{i-j}\right)-cw_n \tag{28}$$

and

$$\chi_m=\begin{cases}0 & m=1,\\ 1 & m>1.\end{cases} \tag{29}$$

Let $w_m^*(\eta)$ denote a special solution of (26) and $\mathcal{L}^{-1}$ the inverse operator of $\mathcal{L}$, respectively. We have

$$w_m^*(\eta)=\chi_m\sum_{n=1}^{m-1}\sigma_{m-n}(c_2)w_n(\eta) +c_0\sum_{n=0}^{m-1}\nu_{m-n}(c_1)\mathcal{L}^{-1}(R_n(\eta)). \tag{30}$$

So the common solution of (26) reads

$$w_m(\eta)=w_m^*(\eta)+C_1\,\mathrm{e}^{-\eta}+C_2\,\mathrm{e}^{-\kappa_1\eta}+C_3\,\mathrm{e}^{\kappa_2\eta}, \tag{31}$$

which contains the unknown a_{m-1}. According to the boundary conditions (27) and the rule of solution expression (9), we have $C_2=C_3=0$. Moreover, the unknown a_{m-1} and C_1 are governed by

$$w_m^*(0)+C_1=0,\;\; w_m^{*\prime}(0)-C_1=0. \tag{32}$$

Hence, the unknown a_{m-1} can be obtained by solving the linear algebraic equation

$$w_m^*(0)+w_m^{*\prime}(0)=0 \tag{33}$$

and thereafter C_1 is given by

$$C_1=-w_m^*(0). \tag{34}$$

In this way, we can derive $w_m(\eta)$ and a_m for $m=0,1,2,3,\ldots$ successively. Then from (2) and (23), we can obtain the travelling-wave solutions of the Kuramoto-Sivashinsky equation. At the Mth-order approximation, we have the analytic solution of (7), namely

$$w(\eta)\approx W_M(\eta)=\sum_{m=0}^{M}w_m(\eta),\;\; a\approx A_M=\sum_{m=0}^{M}a_m. \tag{35}$$

As we know, there is only one unknown convergence-control parameter c_0 in usual HAM [10], and we can determine the possible valid region of c_0 by the so called c_0-curve. But unfortunately it can not tell us which value of c_0 gives the fastest convergent series. However, in the expression of the obtained solution in this paper, there are at most three unknown convergence-control parameters c_0, c_1, and c_2, which can make sure the convergence of the solutions. As shown in [15], we can determined the possible optimal values of convergence-control parameters by minimizing the averaged residual error

$$E_M=\frac{1}{K}\sum_{j=0}^{K}[\mathcal{N}(W_M(j\Delta x),A_M)]^2, \tag{36}$$

where we usually choose $M=15$, $\Delta x=1/2$, and $K=10$ in this paper. These possible optimal convergence-control parameters will overcome the shortcomings mentioned above in usual HAM and may give the fastest convergent series.

3. Comparisons of Different Approaches

In this section, we will give optimal homotopy analysis approaches with different numbers of unknown convergence-control parameters, and compare them in details. For ease of comparison, we suppose $\alpha=\beta=1$, and $k=-1$ as in [14].

3.1. Optimal c_0 in Case of $c_1=c_2=0$

In this case, the method proposed above degenerates into the usual HAM and there is only one unknown convergence-control parameter c_0. In usual HAM, we can investigate the influence of c_0 on the series of a by means of the so-called c_0-curves. As pointed by Liao [10], the valid region of c_0 is a horizontal line segment. Thus, the valid region of c_0 in this example as shown in Figure 2 is $-1.5<c_0<-0.8$. So we can just

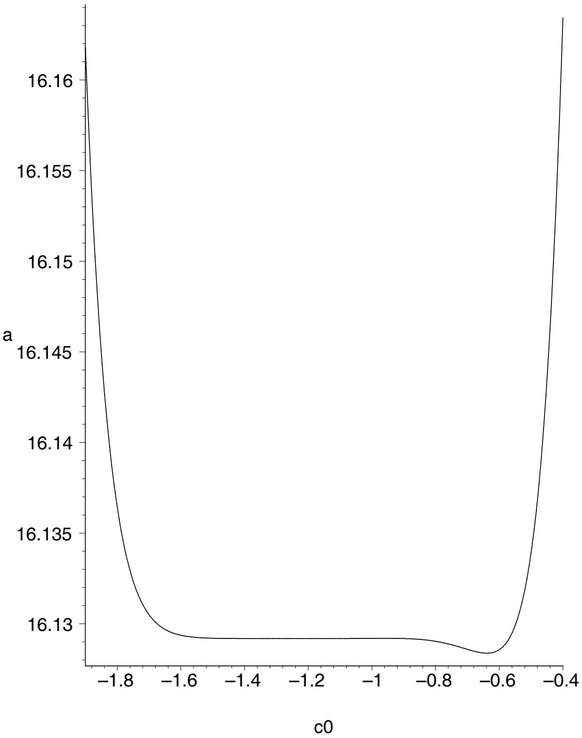

Fig. 2. c_0-curve for the wave amplitude a: 15-order approximation.

determine the possible valid region of c_0. However, c_0-curves usually can not tell us which value of c_0 gives the fastest convergent series and it is a pity that the exact square residual error defined in [15] needs too much CPU time to calculate even if the order of approximation is not very high, and thus is often useless in practice [15].

To overcome this shortcoming, Liao advised to determine the possible optimal value of c_0 by the minimum of averaged residual error E_{10} [15], corresponding to the nonlinear algebraic equation $E'_{10} = 0$. Hence from (6), we have $\mu = 1.875$ for $c = 1.5$. Using the symbolic computation software Maple, by minimizing the averaged residual error (36), we can directly get the optimal convergence-control parameter $c_0 = -1.1175$. According to Table 1, by means of $c_0 = -1.1175$, the value of the residual error converges much faster to 0 than the corresponding homotopy series solution given by usual HAM [14] in case of $c_0 = -1$ and $c_1 = c_2 = 0$, which proves the conclusion drawn by Abbasbandy in [14] that $c_0 = -1$ may not be the best value for the usual HAM. So, even the one-parameter optimal HAM can give much better approximations.

Table 1. Comparison of averaged residual error given by different c_0 in case of $c_1 = c_2 = 0$.

Order of approximation m	Optimal value of c_0	Minimum value of E_m	Value of E_m when $c_0 = -1$
5	-1.1175	1.1565×10^{-6}	2.671×10^{-6}
10	-1.0887	2.4742×10^{-11}	2.5687×10^{-10}
15	-1.1075	1.6871×10^{-12}	3.2283×10^{-12}

Table 2. Comparison of averaged residual error given by different $c_1 = c_2$ in case of $c_0 = -1$.

Order of approximation m	Optimal value of $c_1 = c_2$	Minimum value of E_m	Value of E_m when $c_1 = c_2 = 0$
5	-0.1564	1.1422×10^{-6}	2.671×10^{-6}
10	-0.2442	4.1512×10^{-13}	2.5687×10^{-10}
15	-0.2119	7.787×10^{-17}	3.2283×10^{-12}

3.2. Optimal $c_1 = c_2$ in Case of $c_0 = -1$

Here, we investigate another one-parameter optimal approach in case $c_0 = -1$ with the unknown $c_1 = c_2$. Using the symbolic computation software Maple too, we can directly get the possible optimal convergence-control parameter $c_1 = c_2 = -0.145$. It is found that the homotopy approximations given by $c_0 = -1$ and $c_1 = c_2 = -0.2119$ converges much faster than those given by the usual HAM [14] in case of $c_0 = -1$ and $c_1 = c_2 = 0$, as shown in Table 2. This further illustrates that the second one-parameter optimal HAM is as good as the first one mentioned above.

3.3. Optimal $c_1 \neq c_2$ in Case of $c_0 = -1$

Here, we investigate the two-parameter optimal approach in case $c_0 = -1$ with the unknown $c_1 \neq c_2$. According to above section, we can directly get the optimal convergence-control parameter $c_1 = -0.164$ and $c_2 = -0.154$. As shown in Table 3, it is found that the homotopy approximations given by $c_0 = -1$,

Table 3. Comparison of averaged residual error given by different $c_1 \neq c_2$ in case of $c_0 = 1$.

Order of approximation m	Optimal value of $c_1 \neq c_2$	Minimum value of E_m	Value of E_m when $c_1 = c_2 = 0$
5	$c_1 = -0.07105$, $c_2 = -0.06439$	9.895×10^{-7}	2.671×10^{-6}
10	$c_1 = -0.164$, $c_2 = -0.154$	9.8934×10^{-12}	2.5687×10^{-10}

$c_1 = -0.164$, and $c_2 = -0.154$ converges much faster than those given by the usual HAM [14] in case of $c_0 = -1$ and $c_1 = c_2 = 0$ too. This further proves that the two-parameter optimal homotopy analysis approach is efficient too.

4. Conclusions

In this paper, a solitary wave solution of the Kuramoto-Sivashinsky equation is reconsidered by the optimal HAM. Compared with the usual HAM, more convergence-control parameters are used in the above-mentioned optimal HAM to guarantee the convergence of the homotopy series solution. As shown in this paper, by minimizing the averaged residual error, the possible optimal value of the convergence-control parameters can be obtained which may give the fastest convergent series. Note that the nonlinear operator $\mathcal{N}$ in (20) is rather general so that the above-mentioned optimal HAM can be employed to different types of equations with strong nonlinearity to find the solitary wave solutions with more fast convergence, which we will try in following works.

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